

Non-principal ultrafilters, program extraction and higher order reverse mathematics

Alexander P. Kreuzer

Technische Universität Darmstadt, Germany

Logic Colloquium, Manchester, 2012

Higher order arithmetic

Definition (RCA_0^ω , Kohlenbach '05)

RCA_0^ω is the finite type extension of RCA_0 :

- Sorted into type 0 for \mathbb{N} , type 1 for $\mathbb{N}^{\mathbb{N}}$, type 2 for $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$, \dots ,
- contains basic arithmetic: 0, successor, +, \cdot , λ -abstraction,
- quantifier-free axiom of choice restricted to choice of numbers over functions ($\text{QF-AC}^{1,0}$), i.e.,

$$\forall f^1 \exists y^0 A_{\text{qf}}(f, y) \rightarrow \exists G^2 \forall f^1 A_{\text{qf}}(f, G(f))$$

- and a recursor R_0 , which provides primitive recursion (for numbers),

$$R_0(0, y^0, f) = y, \quad R_0(x + 1, y, f) = f(R_0(x, y, f), x),$$

- Σ_1^0 -induction.

The closed terms of RCA_0^ω will be denoted by T_0 .

Proofs in RCA_0^ω can be simulated in RCA_0 and vice versa.

Arithmetical comprehension

Let Π_1^0 -CA be the schema

$$\forall f \exists g \forall n (g(n) = 0 \leftrightarrow \forall x f(n, x) = 0).$$

Define ACA_0^ω to be $\text{RCA}_0^\omega + \Pi_1^0\text{-CA}$.

Let Feferman's μ be

$$\mu(f) := \begin{cases} \min\{x \mid f(x) = 0\} & \text{if } \exists x f(x) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Denote by (μ) be the statement that μ exists.

Theorem

- $\text{RCA}_0^\omega + (\mu) \vdash \Pi_1^0\text{-CA}$
- $\text{RCA}_0^\omega + (\mu)$ is Π_2^1 -conservative over ACA_0^ω

Theorem (Functional interpretation *relative to μ*)

If

$$\text{RCA}_0^\omega + (\mu) \vdash \forall x \exists y A_{qf}(x, y)$$

the one can extract a term $t \in T_0[\mu]$, such that

$$\text{RCA}_0^\omega + (\mu) \vdash \forall x A_{qf}(x, t(x)).$$

See Avigad, Feferman in Handbook of Proof Theory.

Non-principal ultrafilters

A set $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N})$ is a *non-principal ultrafilter* over \mathbb{N} if

- $\forall X (X \in \mathcal{U} \vee \bar{X} \in \mathcal{U})$,
- $\forall X, Y (X \in \mathcal{U} \wedge X \subseteq Y \rightarrow Y \in \mathcal{U})$,
- $\forall X, Y (X, Y \in \mathcal{U} \rightarrow X \cap Y \in \mathcal{U})$,
- $\forall X (X \in \mathcal{U} \rightarrow X \text{ is infinite})$.

The existence of a non-principal ultrafilter is not provable in ZF.

Non-principal ultrafilters

A set $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N})$ is a *non-principal ultrafilter* over \mathbb{N} if

- $\forall X (X \in \mathcal{U} \vee \bar{X} \in \mathcal{U})$,
- $\forall X, Y (X \in \mathcal{U} \wedge X \subseteq Y \rightarrow Y \in \mathcal{U})$,
- $\forall X, Y (X, Y \in \mathcal{U} \rightarrow X \cap Y \in \mathcal{U})$,
- $\forall X (X \in \mathcal{U} \rightarrow X \text{ is infinite})$.

Coding sets as characteristic function, i.e., $n \in X \equiv [X(n) = 0]$,
this can be formulated in RCA_0^ω :

$$(\mathcal{U}): \left\{ \begin{array}{l} \exists \mathcal{U}^2 (\forall X^1 (X \in \mathcal{U} \vee \bar{X} \in \mathcal{U}) \\ \wedge \forall X^1, Y^1 (X \cap Y \in \mathcal{U} \rightarrow Y \in \mathcal{U}) \\ \wedge \forall X^1, Y^1 (X, Y \in \mathcal{U} \rightarrow (X \cap Y) \in \mathcal{U}) \\ \wedge \forall X^1 (X \in \mathcal{U} \rightarrow \forall n \exists k > n (k \in X)) \\ \wedge \forall X^1 (\mathcal{U}(X) =_0 \text{sg}(\mathcal{U}(X)) =_0 \mathcal{U}(\lambda n. \text{sg}(X(n))))) \end{array} \right.$$

Lower bound on the strength of $\text{RCA}_0^\omega + (\mathcal{U})$

Theorem (K.)

$$\text{RCA}_0^\omega + (\mathcal{U}) \vdash (\mu)$$

In particular, $\text{RCA}_0^\omega + (\mathcal{U})$ proves arithmetical comprehension.

Proof.

Let $f: \mathbb{N} \rightarrow \mathbb{N}$ and set $X_f := \{n \mid \exists m \leq n f(m) = 0\}$.

Then

$$\begin{aligned} \exists n (f(n) = 0) &\iff X_f \text{ is cofinite} \\ &\iff X_f \in \mathcal{U} \end{aligned}$$

Thus

$$\forall f (X_f \in \mathcal{U} \rightarrow \exists n (f(n) = 0 \wedge \forall n' < n f(n') \neq 0))$$

QF-AC^{1,0} yields a functional satisfying (μ) . □

Upper bound on the strength of $\text{RCA}_0^\omega + (\mathcal{U})$

Theorem (K.)

$\text{RCA}_0^\omega + (\mathcal{U})$ is Π_2^1 -conservative over $\text{RCA}_0^\omega + (\mu)$ and thus also over ACA_0^ω .

Proof sketch

Suppose $\text{RCA}_0^\omega + (\mathcal{U}) \vdash \forall f \exists g A(f, g)$ and A does not contain \mathcal{U} .

- 1 The functional interpretation yields a term $t \in T_0[\mu]$, such that

$$\forall f A(f, t(\mathcal{U}, f)).$$

- 2 Normalizing t , such that each occurrence of \mathcal{U} in t is of the form

$$\mathcal{U}(t'(n^0)) \quad \text{for a term } t'(n^0) \in T_0[\mathcal{U}, \mu, f].$$

In particular, \mathcal{U} is only used on **countably many sets** (for each fixed f).

- 3 Build in $\text{RCA}_0^\omega + (\mu)$ a filter which acts on these sets as ultrafilter.

Step 1: Functional interpretation

Suppose $\text{RCA}_0^\omega + (\mathcal{U}) \vdash \forall f^1 \exists g^1 A(f, g)$
where A is arithmetical and does not contain \mathcal{U} .

Modulo μ the formula A is quantifier-free.

Recall (\mathcal{U}) :

$$(\mathcal{U}): \left\{ \begin{array}{l} \exists \mathcal{U}^2 \left(\forall X^1 \left(X \in \mathcal{U} \vee \bar{X} \in \mathcal{U} \right) \right. \\ \quad \wedge \forall X^1, Y^1 \left(X \cap Y \in \mathcal{U} \rightarrow Y \in \mathcal{U} \right) \\ \quad \wedge \forall X^1, Y^1 \left(X, Y \in \mathcal{U} \rightarrow (X \cap Y) \in \mathcal{U} \right) \\ \quad \wedge \forall X^1 \left(X \in \mathcal{U} \rightarrow \forall n \exists k > n (k \in X) \right) \\ \quad \left. \wedge \forall X^1 \left(\mathcal{U}(X) =_0 \text{sg}(\mathcal{U}(X)) =_0 \mathcal{U}(\lambda n. \text{sg}(X(n))) \right) \right) \end{array} \right.$$

Modulo $\text{RCA}_0^\omega + (\mu)$ this is of the form $\exists \mathcal{U}^2 \forall Z^1 (\mathcal{U})_{\text{qf}}(\mathcal{U}, Z)$.

Thus

$$\text{RCA}_0^\omega + (\mu) \vdash \forall \mathcal{U}^2 \forall f^1 \exists Z^1 \exists g^1 \left((\mathcal{U})_{\text{qf}}(\mathcal{U}, Z) \rightarrow A_{\text{qf}}(f, g) \right).$$

Step 1: Functional interpretation (cont.)

$$\text{RCA}_0^\omega + (\mu) \vdash \forall \mathcal{U}^2 \forall f^1 \exists Z^1 \exists g^1 \left((\mathcal{U})_{\text{qf}}(\mathcal{U}, Z) \rightarrow A_{\text{qf}}(f, g) \right).$$

The functional interpretation extracts terms $t_Z, t_g \in T_0[\mu]$, such that

$$\text{RCA}_0^\omega + (\mu) \vdash \forall \mathcal{U}^2 \forall f^1 \left((\mathcal{U})_{\text{qf}}(\mathcal{U}, t_Z(\mathcal{U}, f)) \rightarrow A_{\text{qf}}(f, t_g(\mathcal{U}, f)) \right).$$

Step 2: Term normalization

The terms t_Z, t_g are made of

- 0, successor, +, ·, λ -abstraction
- the primitive recursor R_0 , i.e.

$$R_0(0, y, f) = y, \quad R_0(x + 1, y, f) = f(R_0(x, y, f), x),$$

- μ^2 and
- the parameters \mathcal{U}^2, f^1 .

With coding R_0 is of type 2. The functional \mathcal{U} is also of type 2.

\implies no functional can take \mathcal{U} as parameter.

Lemma

The terms t_Z, t_g can be normalized, such that each occurrence of \mathcal{U} is of the form

$$\mathcal{U}(t'(n^0)) \quad \text{for a term } t' \text{ possible containing } \mathcal{U}, f.$$

Step 3: Construction of (a substitute for) \mathcal{U}

We fix an f and construct a filter \mathcal{F} , such that

$$\text{RCA}_0^\omega + (\mu) \vdash (\mathcal{U})_{qf}(\mathcal{F}, t_Z(\mathcal{F}, f)). \quad (*)$$

This yields then

$$\text{RCA}_0^\omega + (\mu) \vdash \forall f A_{qf}(f, t_g(\mathcal{F}, f))$$

and thus the theorem.

Let t_1, \dots, t_k be the list term with $\mathcal{U}(t_j(n))$ in t_Z, t_g .

- Assume that t_1, \dots is ordered according to the subterm ordering.
- We start with the trivial filter $\mathcal{F}_0 = \{\mathbb{N}\}$.
- For each t_i we build a refined $\mathcal{F}_i \supseteq \mathcal{F}_{i-1}$ such that $(\mathcal{U})_{qf}$ relativized the sets coded by t_1, \dots, t_i holds.
- $\mathcal{F} := \mathcal{F}_k$ solves then $(*)$.

Step 3: Sketch of the construction of \mathcal{F}_1

Let $\mathcal{A} := \{A_1, A_2, \dots\}$ be the set of subsets of \mathbb{N} coded by t_1 .

We assume that \mathcal{A} is closed under union, intersection and inverse.

We want a filter \mathcal{F}_1 , such that

- $\forall X \in \mathcal{A} (X \in \mathcal{F}_1 \vee \overline{X} \in \mathcal{F}_1)$,
- $\forall X, Y \in \mathcal{A} (X \in \mathcal{F}_1 \wedge X \subseteq Y \rightarrow Y \in \mathcal{F}_1)$,
- $\forall X, Y \in \mathcal{A} (X, Y \in \mathcal{F}_1 \rightarrow X \cap Y \in \mathcal{F}_1)$,
- $\forall X \in \mathcal{A} (X \in \mathcal{F}_1 \rightarrow X \text{ is infinite})$.

Construction:

- We decide for each $i = 1, 2, \dots$ whether we put A_i or $\overline{A_i}$ into \mathcal{F}_1 .
- We put A_i into \mathcal{F}_1 if the *intersection of A_i with the previously chosen sets* is infinite. Otherwise we put $\overline{A_i}$ into \mathcal{F}_1 .

Program extraction

Corollary (to the proof)

If $\text{RCA}_0^\omega + (\mathcal{U}) \vdash \forall f \exists g A_{qf}(f, g)$ and A_{qf} does not contain \mathcal{U} then one can extract a term $t \in T_0[\mu]$, such that

$$\text{RCA}_0^\omega + (\mu) \vdash A_{qf}(f, t(f)).$$

Corollary

If $\text{RCA}_0^\omega + (\mathcal{U}) \vdash \forall f \exists g A_{qf}(f, g)$ and A_{qf} does not contain \mathcal{U} then one can extract a term t in **Gödel's System T**, such that

$$A_{qf}(f, t(f))$$

Proof.

- The previous corollary yields a term primitive recursive in μ .
- Interpreting the term using the bar recursor $B_{0,1}$ and then using Howard's ordinal analysis gives a term $t \in T$. □

Possible Applications

Possible Applications:

- Program extraction for ultralimit arguments e.g.,
 - from fixed point theory,
 - Gromov's Theorem.
- Program extraction for non-standard arguments.

- The Π_2^1 -consequences of $\text{RCA}_0^\omega + (\mathcal{U})$ and the Π_2^1 -consequences of ACA_0^ω are the same.
- Program extraction for $\text{RCA}_0^\omega + (\mathcal{U})$.

Thank you for your attention!



Alexander P. Kreuzer

Non-principal ultrafilters, program extraction and higher order reverse mathematics

Journal of Mathematical Logic **12** (2012), no. 1.