Bounded variation and Helly's selection theorem

Alexander P. Kreuzer

ENS Lyon

Universität der Bundeswehr München, October 10th, 2013

Outline

1. Functions of bounded variation
    - Representation
2. Helly's selection theorem

Functions of bounded variation

Definition

- The *variation* of a function $f : [0,1] \to \mathbb{R}$ is defined as follow.

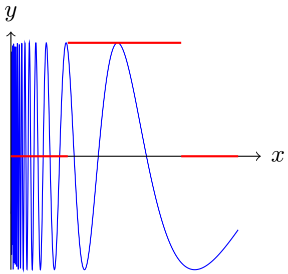
$$V(f) := \sup_{0 \le t_1 < \cdots < t_n \le 1} \sum_{i=1}^{n-1} |f(t_i) - f(t_{i+1})|$$

where $t_1, \ldots, t_n$ ranges over the finite partitions of $[0,1]$.

- f is a *function of bounded variation* if $V(f) < \infty$.

- Examples:
  - Characteristic functions of intervals
  - Continuously differentiable functions.
- Non-example:

$$f(x) = \begin{cases} \sin(1/x) & x > 0, \\ 0 & x = 0. \end{cases}$$



Functions of bounded variation

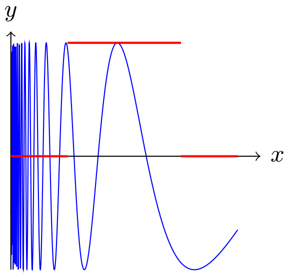
Definition

- The *variation* of a function $f\colon [0,1] \to \mathbb{R}$ is defined as follow.

$$V(f) := \sup_{0 \leq t_1 < \cdots < t_n \leq 1} \sum_{i=1}^{n-1} |f(t_i) - f(t_{i+1})|$$

  where $t_1, \ldots, t_n$ ranges over the finite partitions of $[0,1]$.
- f is a *function of bounded variation* if $V(f) < \infty$.

- There is a correspondence between linear functional on $C([0,1])$ and functions of bounded variation via the Riemann-Stieltjes.



Functions of bounded variation in computable analysis (so far)

Let f be of bounded variation.

Fact

- *f has at most countably many points of discontinuity.*
- *$f_l(x) := \lim_{y \nearrow x} f(y)$ is left-continuous, of bounded variation and $f(x) = f_l(x)$ on all points of continuity.*
- *f and f_l induce the same linear functional on $C([0,1])$.*

- Let x_i be a dense set of points of continuity of f. Represent f by

$$\langle (x_1, f(x_1)), (x_2, f(x_2)), \dots \rangle$$

- f can be recovered by left-continuous extension.
- Successfully applied to give computable interpretation of Jordan decomposition etc. (Weihrauch et. al.)

Functions of bounded variation in computable analysis (so far)

- Left-continuous functions of bounded variation do not form a space.
  - Not closed under taking limits.
- Definition of bounded variation does not generalize to > 1 dimensions.

Sobolev spaces

- The L_1-norm is given by $\|f\|_{L_1} := \int_0^1 |f(x)|\,dx$.
  - The space L_1 is represented as sequences of rational polynomials $\langle p_1, \dots \rangle$ converging at 2^{-n} in L_1-norm.
- The $W^{1,1}$-norm is given by $\|f\|_{W^{1,1}} := \|f\|_{L_1} + \|f'\|_{L_1}$.
  - The derivative f' is taken in the sense of distributions.
  - The space $W^{1,1}$ is represented as sequences of rational polynomials $\langle p_1, \dots \rangle$ converging at 2^{-n} in $W^{1,1}$-norm.

- All $f \in W^{1,1}$ have bounded variation since

$$V(f) = \sup_{0 \le t_1 < \dots < t_n \le 1} \sum_{i=1}^{n-1} |f(t_i) - f(t_{i+1})| = \sup \sum_{i=1}^{n-1} \left| \int_{t_i}^{t_{i+i}} f'\,dx \right|$$
$$\le \sup \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+i}} |f'|\,dx = \int_0^1 |f'|\,dx \le \|f'\|_{W^{1,1}}$$

- Characteristic functions of intervals do not belong to $W^{1,1}$ but have bounded variation.

The space BV

<u>Want:</u> A space BV with

$$L_1 \supseteq BV \supseteq W^{1,1},$$

and variation-norm

$$\|f\|_{BV} = \|f\|_{L_1} + V(f).$$

Problem

Such a space exists, but it is non-separable.

- The family $1_{[0,x]}$ with $x \in \mathbb{R}$ is of the size of the continuum and has mutual distance ≥ 2.

Representation of non-separable spaces. (Brattka)
A point x is represented by

- sequence converging to x (not necessarily at a given rate), and
- norm $v = \|x\|$, or a bounded $v > \|x\|$.

We will use a hybrid approach.

The space BV

The function $f \in BV$ is represent by $\langle v, p_1, p_2, \ldots \rangle$ where

- $\langle p_1, p_2, \ldots \rangle$ represent a function in L_1,
- $v \in \mathbb{Q}$, and
- $\|p_i'\|_1 < v$. (This implies $V(p_i) \leq v$.)

We will call v the *bounded of variation* of f.

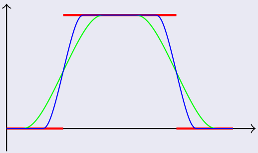
Clear: $L_1 \supseteq BV \supseteq W^{1,1}$

Theorem

For each $f : [0,1] \to \mathbb{R}$ of bounded variation the L_1-equivalence class of f is in BV.

Proof sketch

Approximated
a function of bounded variation f with *mollifications* of f without increasing the variation.



The space BV

The function $f \in BV$ is represent by $\langle v, p_1, p_2, \dots \rangle$ where

- $\langle p_1, p_2, \dots \rangle$ represent a function in L_1,
- $v \in \mathbb{Q}$, and
- $\|p_i'\|_1 < v$. (This implies $V(p_i) \leq v$.)

We will call v the *bounded of variation* of f.

Clear: $L_1 \supseteq BV \supseteq W^{1,1}$

Theorem

For each $f : [0,1] \to \mathbb{R}$ of bounded variation the L_1-equivalence class of f is in BV.

Theorem

For each $f \in BV$ the equivalence class contains a function of bounded variation.

Helly's selection theorem

Theorem (Helly's selection theorem, HST)

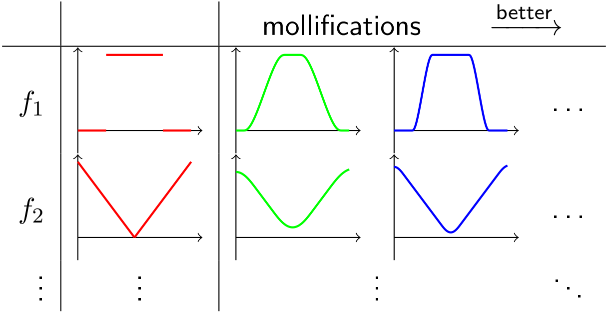
Let $(f_n)_n \subseteq BV$ be a sequence of functions with bounds for variations v_n. If

1. $\|f_n\|_1 \leq u$ *for a* $u \in \mathbb{Q}$,
2. $v_n \leq v$ *for a* $v \in \mathbb{Q}$,

then there exists an $f \in BV$ and a subsequence $f_{g(n)}$ such that $f_{g(n)} \xrightarrow{n \to \infty} f$ in L_1 and the variation of f is bounded by v.

How difficult is it to compute f?

Proof of HST



- If each column of mollifications converges uniformly, then f_i converges in L_1-norm.
- Each column of mollifications is equicontinuous.

$\Rightarrow$ parallelization of Ascoli-Lemma (AA).

- This reduction holds also computationally.
- (Parallelization of) AA can be reduced to (a parallelization of) the Bolzano-Weierstraß principle (BWT). (K. 12)
- (Parallelization of) the BWT can be reduced to a single use of BWT.

Theorem

- HST $\equiv_{\mathrm{W}}$ $\mathsf{BWT}_{\mathbb{R}}$.
- *Over* RCA_0, HST *is instance-wise equivalent to the Bolzano-Weierstraß principle.*

Analysis of Bolzano-Weierstraß principle in the Weihrauch lattice (Brattka, Gherardi, Marcone '12) and (K. '11) for instances of Bolzano-Weierstraß gives the following full classification of HST.

Corollary

- HST $\equiv_{\mathrm{W}}$ WKL$'$
- *Over* RCA_0, HST *is instance-wise equivalent to* WKL *for* Σ^0_1*-trees.*

Summary

- Representation of functions of bounded variation Sobolev-like space.
- Analyzed Helly's selection theorem.

Thank you for your attention!

References I

Vasco Brattka,
Computability on non-separable Banach spaces and Landau's theorem,
From sets and types to topology and analysis, Oxford Logic Guides, vol. **48**, Oxford 2005, 316–333.

Alexander P. Kreuzer,
The cohesive principle and the Bolzano-Weierstraß principle,
Math. Log. Quart. **57** (2011), no. 3, 292–298.

Vasco Brattka, Guido Gherardi, Alberto Marcone,
The Bolzano-Weierstrass theorem is the jump of weak Kőnig's lemma,
Ann. Pure Appl. Logic **163** (2012), no. 6, 623–655.

Alexander P. Kreuzer,
From Bolzano-Weierstraß to Arzelá-Ascoli,
arXiv:1205.5429.

References II

Alexander P. Kreuzer,
Bounded variation and the strength of Helly's selection theorem,
arXiv:1308.3881.