Ramsey's theorem for pairs and program extraction

Alexander P. Kreuzer

TU Darmstadt

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- Ramsey's Theorem for pairs (RT^2_2)
  Every 2-coloring of pairs of $\mathbb{N}$ contains an infinite homogeneous set.
- Each sequence of real numbers contains an infinite monotone subsequence (ADS)
- Each bounded sequence of real numbers contains a slowly converging subsequence ($\mathsf{BW}_{\mathsf{weak}}$).

  (x_n) converges slowly: $\forall k\, \exists n\, \forall m \geq n\ |x_n - x_m| < 2^{-k}$
  (x_n) converges fast: $\forall k\, \forall m \geq k\ |x_k - x_m| < 2^{-k}$

Theorem (K. '10)

$$\mathsf{RCA}_0 \vdash \mathsf{BW}_{\mathsf{weak}} \leftrightarrow \mathsf{COH} + B\Sigma^0_2$$

- $\mathsf{RT}^2_2 \to \mathsf{ADS} \to \mathsf{BW}_{\mathsf{weak}}$
- $\mathsf{RT}^2_2 \not\leftarrow \mathsf{ADS} \not\leftarrow \mathsf{BW}_{\mathsf{weak}}$
  (Hirschfeldt, Shore '07)
- $\mathsf{BW}_{\mathsf{weak}}$ proves $I\Sigma^0_1$ and hence primitive recursion.
- Solutions to computable instances of these principles are in general *not* computable in $0'$.
- Computable instances of these principle have low_2 solutions, i.e. solutions X, such that X'' is computable in $0''$. (Cholak, Jockusch, Slaman '01, Hirschfeldt, Shore '07)

Program extraction

Theorem (K., Kohlenbach '10)

If

$$\mathsf{WKL}_0 + I\Sigma^0_2 + \mathsf{RT}^2_2 \vdash \forall x\, \exists y\, A(x, y),$$

then there exists a term t of Ackermann type, such that

$$\forall x\, A(x, t(x)).$$

Theorem (K., Kohlenbach '10, K. '10)

If

$$\mathsf{WKL}_0 + B\Sigma^0_2 + \mathsf{ADS} + \mathsf{BW}_{\mathsf{weak}} \vdash \forall x\, \exists y\, A(x, y),$$

then there exists a primitive recursive term t, such that

$$\mathsf{PRA} \vdash \forall x\, A(x, t(x)).$$

Motivation

- Proof mining
- Hilbert's program
- Reverse mathematics:
  New proofs for the facts that
  - the functions provable recursive by RT^2_2 are already provably by $I\Sigma^0_2$, cf. Cholak, Jockusch, Slaman '01,
  - ADS and the chain antichain principle does *not* imply $I\Sigma^0_2$, cf. Chong, Slaman, Yang '10.

Sketch of the proof

Theorem

If

$$\mathsf{RCA}_0 + \mathsf{BW}_{\mathsf{weak}} \vdash \forall x\, \exists y\, A(x, y),$$

then there exists a primitive recursive term t, such that

$$\mathsf{PRA} \vdash \forall x\, A(x, t(x)).$$

Sketch of the proof

RCA_0

$$\mathsf{RCA}_0 :\equiv \text{Basic Arithmetic} + \Delta^0_1\text{-CA} + I\Sigma^0_1$$

If $\mathsf{RCA}_0 + \mathsf{WKL} \vdash \forall x\, \exists y\, A(x, y)$,
then exists primitive recursive term t, such that
$\mathsf{PRA} \vdash A(x, t(x))$.

RCA^*_0

$$\mathsf{RCA}^*_0 :\equiv \text{Basic Arithmetic plus } 2^x + \Delta^0_1\text{-CA} + I\Sigma^0_0$$

$$\Pi^0_1\text{-CA}(\phi) :\equiv \exists X\, \forall n\, (n \in X \leftrightarrow \forall y\, \phi(n, y))$$

Theorem (Kohlenbach '98)

If for closed ϕ
$\mathsf{RCA}^*_0 + \mathsf{WKL} + \Pi^0_1\text{-CA}(\phi) \vdash \forall x\, \exists y\, A(x, y)$,
then there exists a primitive recursive term t, such that
$\mathsf{PRA} \vdash A(x, t(x))$.

Sketch of the proof

With this it is sufficient to show

If

$$\mathsf{RCA}_0^* + \mathsf{BW}_{\mathsf{weak}} \vdash \forall x\, \exists y\, A(x,y),$$

then there exists a ϕ, such that

$$\mathsf{RCA}_0^* + \mathsf{WKL} + \Pi^0_1\text{-}\mathsf{CA}(\phi) \vdash \forall x\, \exists y\, A(x,y).$$

There exists a ϕ, such that

$$\mathsf{RCA}_0^* + \mathsf{WKL} + \Pi^0_1\text{-}\mathsf{CA}(\phi[X]) \vdash \exists Y \quad Y \text{ codes a slowly converging subsequence of } X$$

This yields that RCA_0^* + WKL + Π^0_1-CA(ϕ) proves iterations of instances of $\mathsf{BW}_{\mathsf{weak}}$.
Together with a proof- / term-normalization the result follows.

Sketch of the proof

With this it is sufficient to show

If

$$\mathsf{RCA}_0^{*} + \mathsf{BW}_{\mathsf{weak}} \vdash \forall x\, \exists y\, A(x,y),$$

then there exists a ϕ, such that

$$\mathsf{RCA}_0^{*} + \mathsf{WKL} + \Pi^0_1\text{-}\mathsf{CA}(\phi) \vdash \forall x\, \exists y\, A(x,y).$$

For each ψ there exists a ϕ, such that

$$\begin{aligned} \mathsf{RCA}_0^{*} + \mathsf{WKL} + \Pi^0_1\text{-}\mathsf{CA}(\phi[X]) \vdash \\ \exists Y\ \big(Y \text{ codes a slowly converging subsequence of } X \\ \wedge\, \Pi^0_1\text{-}\mathsf{CA}(\psi[X,Y])\big). \end{aligned}$$

This yields that RCA_0^{*} + WKL + Π^0_1-CA(ϕ) proves iterations of instances of $\mathsf{BW}_{\mathsf{weak}}$.

Together with a proof- / term-normalization the result follows.

Proofwise low

Theorem (K', Kohlenbach '10)

Let P *be a principle of the form*

$$\forall X \, \exists Y \, P(X,Y)$$

where P *is* Π^0_3 *and proofwise low over* $\mathsf{RCA}^*_0 + \mathsf{WKL}$*, i.e.*

$$\mathsf{RCA}^*_0 + \mathsf{WKL} + \Pi^0_1\text{-}\mathsf{CA}(\phi[X]) \vdash \exists Y \;\; P(X,Y) \wedge \Pi^0_1\text{-}\mathsf{CA}(\psi[X,Y])\big).$$

If

$$\mathsf{WKL}_0 + B\Sigma^0_2 + \mathsf{P} \vdash \forall x \, \exists y \, A(x,y)$$

then one can extract a primitive recursive term t*, such that*

$$\mathsf{PRA} \vdash \forall x \, A(x, t(x)).$$

Proofwise low (for $I\Sigma^0_2$)

Theorem (K', Kohlenbach '10)

Let P *be a principle of the form*

$$\forall X \, \exists Y \, P(X,Y)$$

where P *is* Π^0_3 *and proofwise low over* WKL_0 *, i.e.*

$$\mathsf{WKL}_0 + \Pi^0_1\text{-}\mathsf{CA}(\phi[X]) \vdash \exists Y \;\; P(X,Y) \\ \wedge \Pi^0_1\text{-}\mathsf{CA}(\psi[X,Y])\big).$$

If

$$\mathsf{WKL}_0 + I\Sigma^0_2 + \mathsf{P} \vdash \forall x \, \exists y \, A(x,y)$$

then one can extract a term t *of Ackermann type, such that*

$$\forall x \, A(x,t(x)).$$

Summary of the strength of the Bolzano-Weierstraß principle

- The Bolzano-Weierstraß principle implies ACA_0 and hence PA. In general *no* primitive recursive terms. (Friedman '76)
- Sequences of *instances* of the Bolzano-Weierstraß principle over $\mathrm{RCA}_0^* + \mathrm{WKL}$ allow extraction of primitive recursive terms. (Kohlenbach's elimination of Skolemfunctions for monotone formulas '98)
- We showed that general uses of the weak Bolzano-Weierstraß principle and ADS even over $\mathrm{RCA}_0 + \mathrm{WKL}$ allow extraction of primitive recursive terms.

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