NON-PRINCIPAL ULTRAFILTERS, PROGRAM EXTRACTION AND HIGHER ORDER REVERSE MATHEMATICS

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Abstract. We investigate the strength of the existence of a non-principal ultrafilter over fragments of higher order arithmetic.

Let $(\mathcal{U})$ be the statement that a non-principal ultrafilter on $\mathbb{N}$ exists and let ACA_0^ω be the higher order extension of ACA_0. We show that $\mathsf{ACA}_0^\omega + (\mathcal{U})$ is Π^1_2-conservative over ACA_0^ω and thus that $\mathsf{ACA}_0^\omega + (\mathcal{U})$ is conservative over PA.

Moreover, we provide a program extraction method and show that from a proof of a strictly Π^1_2 statement $\forall f\, \exists g\, \mathsf{A}_{qf}(f,g)$ in $\mathsf{ACA}_0^\omega + (\mathcal{U})$ a realizing term in Gödel's system T can be extracted. This means that one can extract a term $t \in T$, such that $\forall f\, \mathsf{A}_{qf}(f, t(f))$.

In this paper we will investigate the strength of the existence of a non-principal ultrafilter over fragments of higher order arithmetic. We will classify the consequences of this statement in the spirit of reverse mathematics. Furthermore, we will provide a program extraction method.

Let $(\mathcal{U})$ be the statement that a non-principal ultrafilter on $\mathbb{N}$ exists. Let RCA_0^ω, ACA_0^ω be the extensions of RCA_0 resp. ACA_0 to higher order arithmetic as introduced by Kohlenbach in [16]. In RCA_0^ω or ACA_0^ω the statement $(\mathcal{U})$ can be formalized using an object of type $\mathbb{N}^\mathbb{N} \longrightarrow \mathbb{N}$.

Further, let Feferman's μ be a functional of type $\mathbb{N}^\mathbb{N} \longrightarrow \mathbb{N}$ satisfying

$$f(\mu(f)) = 0 \quad \text{if} \quad \exists x\, f(x) = 0$$

and let (μ) be the statement that such a functional exists. It is clear that (μ) implies arithmetical comprehension. However, μ is not definable in ACA_0^ω.

We will show that

- over RCA_0^ω the statement $(\mathcal{U})$ implies (μ) and therefore is strictly stronger than ACA_0^ω, and that
- $\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U})$ is Π^1_2-conservative over ACA_0^ω and therefore also conservative over PA. Moreover, we will show that from a proof of $\forall f\, \exists g\, \mathsf{A}_{qf}(f,g)$ in $\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U})$, where A_{qf} is quantifier free, one can extract a realizing term t in Gödel's system T, i.e. a term such that $\forall f\, \mathsf{A}_{qf}(f, t(f))$.

The system $\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U})$ is strong in the sense that one can carry out many common ultralimit and non-standard arguments. For instance one can carry out in this theory the construction of Banach limits and many Loeb measure constructions.

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Our result shows that this system is weak with respect to Π^1_2 sentences. Moreover, our program extraction result shows that one can still obtain constructive (even primitive recursive in the sense of Gödel) realizers and bounds from proofs using highly non-constructive objects like non-principal ultrafilters.

Using this technique it is possible to extract bounds from proofs using ultralimits and non-standard techniques. Such proofs do occur in mathematics, for instance in metric fixed point theory, see [1] and [12]. In [8], Gerhardy extracted a rate of proximity from such a proof by eliminating the ultrafilter by hand. Our result here shows that this can be done with similar uses of ultrafilters.

Comparison with other approaches. Solovay constructed a filter which acts on the hyperarithmetical sets like a non-principal ultrafilter. With this he showed an effective version of the Galvin-Prikry theorem, see [21]. His construction of the partial ultrafilter is similar to ours. Avigad analyzed his result in terms of reverse mathematics and formalized this particular proof in ATR_0, see [2]. However, this result does not follow from our meta-theorem, since it not only uses a non-principal ultrafilter but also a substantial amount of transfinite recursion.

Using our approach one also obtains upper bounds on the strength of non-standard analysis and program extraction methods. This can be done by constructing an ultrapower model of non-standard analysis in $\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U})$. If one is not interested in the ultrafilter but only in the axiomatic treatment of non-standard analysis one can obtain refined results by interpreting it directly, see for instance [3], [11] and for program extraction [5].

Palmgren used in [19] an approach similar to ours to interpret non-standard arithmetic. He builds (partial) non-principal ultrafilters for the definable sets of a fixed level in the arithmetical hierarchy and obtains a conservation results similar to ours. However he cannot treat ultrafilter nor obtains program extraction.

In reverse mathematics idempotent ultrafilters are considered in the context of Hindman's theorem, which can be proven using an idempotent ultrafilter (or at least a countable part of it), see Hirst [10] and Towsner [23]. We code an ultrafilter over $\mathbb{N}$ like Hirst does. However, our construction of ultrafilters is different since we are not aiming for idempotent ultrafilters. An idempotent ultrafilter is a very special ultrafilter and it seems that even the construction of countable parts of an idempotent ultrafilter requires a system that is proof theoretically stronger than $\mathsf{ACA}_0^\omega + (\mu)$ and is therefore beyond our method.

Recently, Towsner considered the addition of an ultrafilter to fragments of second order arithmetic. He adds the ultrafilter as a predicate over sets. Independently he also obtains conservation results for non-principal ultrafilter related to ours but using different methods. However, in the context he considers, the non-principal ultrafilter is weaker in the sense that one cannot use it to define other higher-order objects like μ and thus cannot use the ultrafilter to prove statements beyond ACA_0. See [22].

Enayat considered which kind of non-principal ultrafilters can be defined on the second order part of models of say ACA_0, see [6].

Logical system. We will work in fragments of Peano arithmetic in all finite types. The set of all finite types $\mathbf{T}$ is defined to be the smallest set that satisfies

$$0 \in \mathbf{T}, \qquad \rho, \tau \in \mathbf{T} \Rightarrow \tau(\rho) \in \mathbf{T}.$$

The type 0 denotes the type of natural numbers and the type $\tau(\rho)$ denotes the type of functions from ρ to τ. The type $0(0)$ is abbreviated by 1 the type $0(0(0))$ by 2. The degree of a type is defined by

$$deg(0) := 0 \qquad deg(\tau(\rho)) := \max(deg(\tau), deg(\rho) + 1).$$

The type of a variable will sometimes be written as superscript of a term.

Equality $=_0$ for type 0 objects will be added as primitive notion to the systems together with the usual equality axioms. Higher type equality $=_{\tau\rho}$ will be treated as abbreviation:

$$x^{\tau\rho} =_{\tau\rho} y^{\tau\rho} :\equiv \forall z^\rho \, xz =_\tau yz.$$

Define the λ-combinators $\Pi_{\rho,\sigma}, \Sigma_{\rho,\sigma,\tau}$ for $\rho, \sigma, \tau \in \mathbf{T}$ to be the functionals satisfying

$$\Pi_{\rho,\sigma} x^\rho y^\sigma =_\rho x, \qquad \Sigma_{\rho,\sigma,\tau} x^{\tau\sigma\rho} y^{\sigma\rho} z^\rho =_\tau xz(yz).$$

Similar define the recursor R_ρ of type ρ to be the functional satisfying

$$R_\rho 0yz =_\rho y, \qquad R_\rho(Sx^0)yz =_\rho z(R_\rho xyz)x.$$

Let *Gödel's system* T be the $\mathbf{T}$-sorted set of closed terms that can be build up from 0^0, the successor function S^1, the λ-combinators and, the recursors R_ρ for all finite types ρ. Using the λ-combinators one easily sees that T is closed under λ-abstraction, see [24]. Denote by T_0 the subsystem of Gödel's system T, where primitive recursion is restricted to recursors R_0. The system T_0 corresponds to the extension of Kleene's primitive recursive functionals to mixed types, see [13], whereas full system T corresponds to Gödel's primitive recursive functionals, see [9]. By $T_0[F]$ we will denote the system resulting from adding a function(al) F to T_0.

The system RCA_0^ω is defined to be the extension of the term system T_0 by Σ^0_1-induction, the extensionality axioms

$$(\mathsf{E}_{\rho,\tau})\colon \forall z^{\tau\rho}, x^\rho, y^\rho \, (x =_\rho y \rightarrow zx =_\tau zy)$$

for all $\tau, \rho \in \mathbf{T}$, and the schema of quantifier free choice restricted to choice of numbers over functions ($\mathsf{QF\text{-}AC}^{1,0}$), i.e.

$$\forall f^1 \, \exists x^0 \, \mathsf{A}_{qf}(f, x) \rightarrow \exists F^2 \, \forall f^1 \, \mathsf{A}_{qf}(f, F(f)).$$

This schema is the higher order equivalent to recursive comprehension (Δ^0_1-CA). (Strictly speaking the system RCA_0^ω was defined in [16] to contain only quantifier free induction instead of Σ^0_1-induction. Since Σ^0_1-induction is provable in that system, we may also add it directly.) The systems WKL_0^ω, ACA_0^ω are defined to be $\mathsf{RCA}_0^\omega + \mathsf{WKL}$ resp. $\mathsf{RCA}_0^\omega + \Pi^0_1$-$\mathsf{CA}$.

All of these systems are conservative over their second-order counterparts, where the second-order part is given by functions instead of sets. These second-order systems can then be interpreted in RCA_0, resp. WKL_0, ACA_0. See [16].

The system RCA_0^ω has a functional interpretation (always combined with elimination of extensionality and a negative translation) in T_0. The system ACA_0^ω has a functional interpretation in $T_0[\mu]$ if one interprets comprehension using μ or in $T_0[B_{0,1}]$ if one interprets comprehension using the bar recursor of lowest type $B_{0,1}$. See [16] and [4] for the interpretation using μ and [17, Section 11] for the interpretation using $B_{0,1}$. For a general survey on the functional interpretation see [17] and [4].

Definition 1 (non-principal ultrafilter, $(\mathcal{U})$)**.** Let $(\mathcal{U})$ be the statement that there exists a non-principal ultrafilter (on $\mathbb{N}$):

$$(\mathcal{U})\colon \begin{cases} \exists \mathcal{U}^2 \big(\, \forall X \, \big(X \in \mathcal{U} \lor \overline{X} \in \mathcal{U}\big) \\ \qquad \land \forall X^1, Y^1 \, (X \cap Y \in \mathcal{U} \to Y \in \mathcal{U}) \\ \qquad \land \forall X^1, Y^1 \, (X, Y \in \mathcal{U} \to (X \cap Y) \in \mathcal{U}) \\ \qquad \land \forall X^1 \, (X \in \mathcal{U} \to \forall n \, \exists k > n \, (k \in X)) \\ \qquad \land \forall X^1 \, (\mathcal{U}(X) =_0 \mathrm{sg}(\mathcal{U}(X)) =_0 \mathcal{U}(\lambda n.\, \mathrm{sg}(X(n)))) \, \big) \end{cases}$$

Here $X \in \mathcal{U}$ is an abbreviation for $\mathcal{U}(X) =_0 0$. The type 1 variables X, Y are viewed as characteristic functions of sets, where $n \in X$ is defined to be $X(n) = 0$. The operation $\cap$ is defined as taking the pointwise maximum of the characteristic functions. With this the intersection of two sets can be expressed in a quantifier free way. The last line of the definition states that $\mathcal{U}$ yields the same value for different characteristic functions of the same set and that $\mathcal{U}(X) \leq 1$.

For notational ease we will usually add a Skolem constant $\mathcal{U}$ and denote this also with $(\mathcal{U})$.

The second line in the definition of $(\mathcal{U})$ is equivalent to the following axiom usually found in the axiomatization of (ultra)filters:

$$\forall X, Y \, (X \subseteq Y \land X \in \mathcal{U} \to Y \in \mathcal{U}) \, .$$

We avoided this statement in $(\mathcal{U})$ since $\subseteq$ cannot be expressed in a quantifier free way.

Lemma 2 (finite partition property)**.** *The ultrafilter $\mathcal{U}$ satisfies the finite partition property over RCA_0^ω. This means that for each finite partition $(X_i)_{i<n}$ of $\mathbb{N}$ the system RCA_0^ω proves that there exists a unique $i < n$ with $X_i \in \mathcal{U}$.*

Proof. We prove by quantifier free induction on m the statement

$$\exists! i \leq m \left(\Big(i < m \to X_i \in \mathcal{U}\Big) \land \ \Big(i = m \to \bigcup_{j=m}^{n-1} X_j \in \mathcal{U}\Big)\right). \tag{1}$$

In the cases $m < 2$ the statement follows directly from $(\mathcal{U})$. For the induction step we assume that the statement for m holds. This means there exists an i as stated in (1). If $i < m$ then this i also satisfies (1) with m replaced by $m + 1$ and we are done. Otherwise we have $\bigcup_{j=m}^{n-1} X_j \in \mathcal{U}$.

The axiom $(\mathcal{U})$ yields

$$\bigcup_{j=0}^{m} X_j \in \mathcal{U} \ \lor \bigcup_{j=m+1}^{n-1} X_j \in \mathcal{U}.$$

If the left side of the disjunction holds then

$$X_m = \bigcup_{j=0}^{m} X_j \cap \bigcup_{j=m}^{n-1} X_j \in \mathcal{U}$$

and $i := m$ satisfies the (1) with m replaced by $m + 1$. If the right side of the disjunction holds $i := m + 1$ satisfies (1).

The lemma follows from (1) by taking $m := n$. ∎

Theorem 3.

$$\mathsf{RCA}_0^\omega + (\mathcal{U}) \vdash (\mu)$$

In particular, $\mathsf{RCA}_0^\omega + (\mathcal{U})$ *is strictly stronger than* ACA_0^ω*.*

Proof. Let $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ be a function. The set $X_f := \{x \in \mathbb{N} \mid \exists x' < x\, f(x') = 0\}$ is cofinal if $\exists x\, f(x) = 0$, if not then the set X_f is empty. Hence

$$X_f \in \mathcal{U} \quad \text{iff} \quad \exists x\, f(x) = 0.$$

From this it follows that

$$\forall f\, \exists x\, (X_f \in \mathcal{U} \rightarrow f(x) = 0)\,.$$

An application of $\mathsf{QF\text{-}AC}^{1,0}$ now yields a functional satisfying (μ). ☐

Theorem 4 (Program extraction)**.** *Let* $\mathsf{A}_{qf}(f,g)$ *be a quantifier free formula of* RCA_0^ω *containing only* f, g *free. In particular,* A_{qf} *must not contain* μ *or* $\mathcal{U}$*.*

If

$$\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U}) \vdash \forall f^1\, \exists g^1\, \mathsf{A}_{qf}(f,g)$$

then one can extract a closed term $t \in T$ *such that*

$$\forall f\, \mathsf{A}_{qf}(f, tf).$$

The proof of this theorem proceeds in five steps:

1. Using the functional interpretation and proof theoretic methods developed in [18] we show that a proof of the statement
$$\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U}) \vdash \forall f\, \exists g\, \mathsf{A}_{qf}(f,g)$$
can be normalized in such a way that each application of the functional $\mathcal{U}$ that occurs in the proof has the form $\mathcal{U}(t[n^0])$, where t is a term that contains only n free and with $\lambda n.t \in T_0[\mathcal{U}]$. (We do not have to consider μ here, since it can be defined from $\mathcal{U}$ by Theorem 3.) In particular, this shows that the ultrafilter $\mathcal{U}$ is used only on countably many sets.
2. We show that we can construct in $\mathsf{RCA}_0^\omega + (\mu)$ a *partial ultrafilter*, that is an object that behaves like an ultrafilter on the sets that occur in the proof. We then replace $\mathcal{U}$ by this partial ultrafilter and obtain a proof of $\forall f\, \exists g\, \mathsf{A}_{qf}(f,g)$ in $\mathsf{RCA}_0^\omega + (\mu)$.
3. By [4, 7], the theory $\mathsf{RCA}_0^\omega + (\mu)$ is conservative over ACA_0^ω for such sentences. Thus, we obtain $\mathsf{ACA}_0^\omega \vdash \forall f\, \exists g\, \mathsf{A}_{qf}(f,g)$.
4. Applying the functional interpretation to this statement and interpreting the comprehension using $B_{0,1}$ yields a term $t^2 \in T_0[B_{0,1}]$, such that
$$\forall f\, \mathsf{A}_{qf}(f, tf).$$
5. Since this term t is only of type 2, one can use an ordinal analysis of the bar recursor to eliminate it and obtain a new term $t' \in T$, such that $t' =_2 t$ and hence that
$$\forall f\, \mathsf{A}_{qf}(f, t'f).$$

Before we prove this theorem we show how to construct a partial ultrafilter and provide some proof theoretic lemmata.

Partial ultrafilter.

Definition 5 (partial ultrafilter)**.**

- Call a set $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ of subsets of natural numbers, that is closed under complement, finite unions and finite intersections, an *algebra*.
- Let $\mathcal{A}$ be an algebra. Call a set $\mathcal{F} \subseteq \mathcal{A}$ a *partial non-principal ultrafilter* for $\mathcal{A}$ iff $\mathcal{F}$ satisfies the non-principal ultrafilter axioms in Definition 1 relativized to $\mathcal{A}$, i.e.

$$\begin{cases} \quad \forall X \in \mathcal{A} \left(X \in \mathcal{F} \vee \overline{X} \in \mathcal{F} \right) \\ \wedge \forall X, Y \in \mathcal{A} \, (X \cap Y \in \mathcal{F} \rightarrow Y \in \mathcal{F}) \\ \wedge \forall X, Y \in \mathcal{A} \, (X, Y \in \mathcal{F} \rightarrow (X \cap Y) \in \mathcal{F}) \\ \wedge \forall X \in \mathcal{A} \, (X \in \mathcal{F} \rightarrow \forall n \, \exists k > n \, k \in X) \\ \wedge \forall X^1 \, (\mathcal{F}(X) =_0 \mathrm{sg}(\mathcal{F}(X)) =_0 \mathcal{F}(\lambda n. \, \mathrm{sg}(X(n)))\,. \end{cases}$$

The sets $\mathcal{A}$ and $\mathcal{F}$ are given here—like $\mathcal{U}$—as characteristic functions. In the following we will also refer to algebras and filters given by a countable sequence of sets, i.e. $\mathcal{A} = (A_i)_{i \in \mathbb{N}}$ resp. $\mathcal{F} = (F_i)_{i \in \mathbb{N}}$. In this case the characteristic function $\chi_{\mathcal{A}}$ of $\mathcal{A}$ can be defined using μ. It is given by

$$\chi_{\mathcal{A}} = \begin{cases} 0 & \text{if } \exists i \ (A_i = B), \\ 1 & \text{otherwise,} \end{cases}$$

where the set equality $A_i = B$ is definable using μ. The characteristic function for $\mathcal{F}$ is defined likewise.

Note that in RCA_0^ω every sequence of sets can be extended to a countable algebra. Further note that a partial non-principal ultrafilters for countable algebras are also countable.

A partial ultrafilter $\mathcal{F}$ can be viewed as the closed subset $\{\mathcal{U} \in \beta\mathbb{N} \mid \mathcal{U} \supseteq \mathcal{F}\}$ of the Stone-Čech compactification $\beta\mathbb{N}$.

Proposition 6. *Let $\mathcal{A}$ be a countable algebra and let $\mathcal{F} = (F_i)_{i \in \mathbb{N}}$ be a countable partial non-principal ultrafilter for $\mathcal{A}$. Then $\mathsf{RCA}_0^\omega + (\mu)$ proves that for each countable extension $\tilde{\mathcal{A}} = (\tilde{A}_i)_{i \in \mathbb{N}} \supseteq \mathcal{A}$ there exists a countable partial non-principal ultrafilter $\tilde{\mathcal{F}} = (\tilde{F}_i)_{i \in \mathbb{N}} \supseteq \mathcal{F}$.*

Proof. In the following let x be the code for a tuple $\langle x_0, \dots, x_{\mathrm{lth}(x)-1} \rangle$ in $2^{<\mathbb{N}}$. Let

$$\tilde{A}^x := \bigcap_{i < \mathrm{lth}\, x} \begin{cases} \tilde{A}_i & \text{if } x_i = 0, \\ \overline{\tilde{A}_i} & \text{if } x_i = 1. \end{cases}$$

Using quantifier free induction one easily sees that for every n the set $\left\{ \tilde{A}^x \;\middle|\; x \in 2^n \right\}$ defines a partition of $\mathbb{N}$, i.e. for all z

$$\forall n \, \exists! x \in 2^n \, \left(z \in \tilde{A}^x \right). \tag{2}$$

Let $T(x)$ be the Π^0_2-predicate given by

$$\forall j \, \left(\tilde{A}^x \cap F_j \text{ is infinite} \right).$$

Note that $T(x)$ defines a tree on $2^{<\mathbb{N}}$. This tree is infinite because otherwise we would have

$$\exists n \, \forall x \in 2^n \, \exists j \, \exists y \, \forall z > y \, \left(z \notin \tilde{A}^x \cap F_j \right).$$

The bounded collection principle Π^0_1-CP then yields

$$\exists n\, \exists j^*, y^*\, \forall x \in 2^n\, \forall z > y^* \left(z \notin \tilde{A}^x \cap \bigcap_{j \leq j^*} F_j \right). \tag{3}$$

The set $\bigcap_{j \leq j^*} F_j$ is in $\mathcal{F}$ and is, therefore, infinite. In particular, it contains an element z which is bigger than y^*. Because $\tilde{A}^x$ with $x \in 2^n$ defines a partition of $\mathbb{N}$ there is an x such that $z \in \tilde{A}^x$. This contradicts (3) and therefore the tree T is infinite.

Hence we obtain using Π^0_2-WKL, i.e. weak König's Lemma for trees given by a Π^0_2-statement, an infinite branch b of T. Note that Π^0_2-WKL is provable in ACA^ω_0. We claim that the set

$$\tilde{\mathcal{F}} = \left\{ \tilde{A}_i \mid b(i) = 0 \right\}$$

defines then a partial non-principal ultrafilter for $\tilde{\mathcal{A}}$ which contains $\mathcal{F}$.

It is clear that each set $\tilde{A}_i \in \tilde{\mathcal{F}}$ is infinite since $\tilde{A}^{\overline{b}(i+1)}$ is a subset of $\tilde{A}_i$ and is infinite by definition of the tree. For each set $\tilde{A}_i$ exactly one of $\tilde{A}_i$, $\overline{\tilde{A}_i} = \tilde{A}_j$ is in $\tilde{\mathcal{F}}$ since otherwise the set $\tilde{A}^{\overline{b}(\max(i,j)+1)}$ is empty and therefor not infinite, which contradicts the definition of the tree. Now suppose $\tilde{A}_i, \tilde{A}_{i'} \in \tilde{\mathcal{F}}$ then the intersection $\tilde{A}_j$ is also in $\tilde{\mathcal{F}}$ since $\tilde{A}^{\overline{b}(\max(i,i',j)+1)} \subseteq \tilde{A}_i \cap \tilde{A}_{i'} = \tilde{A}_j$, which rules out the case that $\overline{\tilde{A}_j}$ is in $\tilde{\mathcal{F}}$. For a similar reason also supersets of set in $\tilde{\mathcal{F}}$ are also in the filter.

Each set $\tilde{A}_i = F_i \in \mathcal{F}$ is also in $\tilde{\mathcal{F}}$ since otherwise the complement $\overline{\tilde{A}_i} = \tilde{A}_j$ would be in $\mathcal{F}$ and then $\tilde{A}^{\overline{b}(j+1)} \subseteq \tilde{A}_j$ which has empty intersection with F_i and this contradicts the definition of the tree. □

Proof theory. The system RCA^ω_0 contains full extensionality. Since extensionality cannot be expressed in a purely universal statement it contains some constructive content. For this reason the functional interpretation cannot handle this general form of extensionality directly and it has to be eliminated beforehand. The system RCA^ω_0 is formulated in a way that this can be done using standard methods, i.e. the elimination of extensionality, see for instance [17, Section 10.4]. Since we added a new higher order constant $\mathcal{U}$ we have to check manually that this constant is extensional. This will be done in the following lemma. To formulate it we will need a *weakly extensional system*, i.e. a system in which extensionality is restricted to a rule of extensionality that only allows quantifier free premises. We will use $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{1,0}$. This system is the weakly extensional counterpart to RCA^ω_0 in the sense that RCA^ω_0 results from $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{1,0}$ by adding the extensionality axioms. (In other words $\mathsf{RCA}^\omega_0 \equiv \widehat{\mathsf{E\text{-}PA}}^\omega\!\restriction + \mathsf{QF\text{-}AC}^{1,0}$.)

Lemma 7 (Elimination of extensionality)**.** *The system* $\widehat{\mathsf{WE\text{-}PA}}^\omega\!\restriction + (\mathcal{U})$ *proves that* $\mathcal{U}$ *is extensional, i.e.*

$$\forall X, Y \left(\forall k\ (k \in X \leftrightarrow k \in Y) \to (X \in \mathcal{U} \leftrightarrow Y \in \mathcal{U})\right).$$

In particular, the elimination of extensionality is applicable to $\mathsf{RCA}^\omega_0 + (\mathcal{U})$*. This means the following rule holds: If* A *is a sentence that contains only quantification over variables of degree* ≤ 1 *and*

$$\mathsf{RCA}^\omega_0 \vdash (\mathcal{U}) \to \mathsf{A}$$

then

$$\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction} + \mathsf{QF\text{-}AC}^{1,0} \vdash (\mathcal{U}) \rightarrow \mathsf{A}.$$

Proof. Suppose that $\mathcal{U}$ is not extensional. Then there exist two sets X, Y, such that

$$\forall k\ (k \in X \leftrightarrow k \in Y)) \quad \text{and} \quad X \in \mathcal{U} \wedge Y \notin \mathcal{U}.$$

By the axiom $(\mathcal{U})$ we obtain that $\overline{Y} \in \mathcal{U}$ and with this

$$X \cap \overline{Y} \in \mathcal{U}.$$

By the second to last line of $(\mathcal{U})$ there exists an $n \in X \cap \overline{Y}$. This contradicts the assumption and we conclude that $\mathcal{U}$ is extensional.

For the elimination of extensionality we use the techniques presented in Section 10.4 of [17]. We will also use the notation introduced in this section for the rest of this proof: The extensionality of $\mathcal{U}$ translates into $\mathcal{U} =^e \mathcal{U}$. Since $(\mathcal{U})$ contains (after the Skolemization) only quantification degree ≤ 1 and the constant $\mathcal{U}$ is extensional, we obtain $(\mathcal{U})_e \leftrightarrow (\mathcal{U})$. Because A does not contain quantification of degree > 1 we also obtain that A_e is equivalent to A. Hence $(\mathcal{U}) \rightarrow \mathsf{A}$ does not change under the $(\cdot)_e$ relativization.

The lemma now follows from Proposition 10.45 in [17] relativized according to [17, Section 10.5] to RCA_0^ω. ☐

The next theorem will provide the term normalization that is needed for the proof of Theorem 4.

Theorem 8 (term-normalization for degree 2)**.** *Let $F_1, \dots, F_n$ be constants of degree ≤ 2.*

For every term $t^1 \in T_0[F_1, \dots, F_n]$ there is a term $\tilde{t} \in T_0[F_0, \dots, F_{n-1}]$ with

$$\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction} \vdash t =_1 \tilde{t}$$

and such that every occurrence of an F_i in $\tilde{t}$ is of the form

$$F_i(\tilde{t}_0[y^0], \dots, \tilde{t}_{k-1}[y^0]).$$

Here k is the arity of F_i, and $\tilde{t}_j[y^0]$ are fixed terms whose only free variable is y^0.

Proof. See Theorem 20 in [18]. For a reference see also [15, proof of Proposition 4.2]. This normalization is similar to the normalization described in Section 8.3 of [4]. ☐

The axiom $(\mathcal{U})$ can be prenexted into a statement of the form

$$\begin{aligned} \exists \mathcal{U}^2\, \forall X^1, Y^1\, \forall n\, \exists k\ \big(\quad & \big(X \in \mathcal{U} \vee \overline{X} \in \mathcal{U}\big) \\ \wedge\ & (X \cap Y \in \mathcal{U} \rightarrow Y \in \mathcal{U}) \\ \wedge\ & (X, Y \in \mathcal{U} \rightarrow (X \cap Y) \in \mathcal{U}) \\ \wedge\ & (X \in \mathcal{U} \rightarrow (k > n \wedge k \in X)) \\ \wedge\ & (\mathcal{U}(X) =_0 \mathrm{sg}(\mathcal{U}(X)) =_0 \mathcal{U}(\lambda n.\, \mathrm{sg}(X(n))))\big). \end{aligned}$$

By coding the sets X, Y together into one set Z and calling the quantifier free matrix of the above statement $(\mathcal{U})_{qf}$ we arrive at

$$\exists \mathcal{U}^2\, \forall Z^1\, \forall n\, \exists k\, (\mathcal{U})_{qf}(\mathcal{U}, Z, n, k).$$

Applying $\mathsf{QF\text{-}AC}^{1,0}$ yields

$$\exists \mathcal{U}^2\, \exists K^2\, \forall Z^1\, \forall n\, (\mathcal{U})_{qf}(\mathcal{U}, Z, n, KnZ). \tag{4}$$

Note that $\mathcal{U}$ and K are only of degree 2. This will be crucial for the following proof.

For K one may always choose

$$K'(n,X) := \begin{cases} \min\{k \in X \mid k > n\} & \text{if exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The functional K' is definable using μ. Therefore the real difficulty lies in finding a solution for $\mathcal{U}$.

We are now in the position to give a proof of Theorem 4.

Proof of Theorem 4. In the light of Theorem 3 suffices to consider only $\mathsf{RCA}_0^\omega + (\mathcal{U})$ instead of $\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U})$.

Let $\mathsf{A}_{qf}(f,g)$ be a quantifier-free statement not containing $\mathcal{U}$, such that

$$\mathsf{RCA}_0^\omega + (\mathcal{U}) \vdash \forall f^1\, \exists g^1\, \mathsf{A}_{qf}(f,g).$$

By the deduction theorem we obtain

$$\mathsf{RCA}_0^\omega \vdash (\mathcal{U}) \to \forall f\, \exists g\, \mathsf{A}_{qf}(f,g).$$

Using Lemma 7 we obtain

$$\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction} + \mathsf{QF\text{-}AC}^{1,0} \vdash (\mathcal{U}) \to \forall f\, \exists g\, \mathsf{A}_{qf}(f,g).$$

Reintroducing a variable $\mathcal{U}$ for the ultrafilter together with (4) gives

$$\big(\exists \mathcal{U}^2\, \exists K^2\, \forall Z^1\, \forall n\, (\mathcal{U})_{qf}(\mathcal{U}, Z, n, KnZ)\big) \to \forall f\, \exists g\, \mathsf{A}_{qf}(f,g)$$

which is equivalent to

$$\forall f\, \forall \mathcal{U}^2\, \forall K^2\, \exists Z^1, n\, \exists g\, \left((\mathcal{U})_{qf}(\mathcal{U}, Z, n, KnZ) \to \mathsf{A}_{qf}(f,g)\right).$$

A functional interpretation yields terms $t_Z, t_n, t_g \in T_0[\mathcal{U}, K, f]$ such that

$$\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction} \vdash \forall f\, \forall \mathcal{U}^2\, \forall K^2\, \left((\mathcal{U})_{qf}(\mathcal{U}, t_Z, t_n, K t_n t_Z) \to \mathsf{A}_{qf}(f, t_g)\right); \tag{6}$$

see for instance Theorem 10.53 in [17]. Now by Theorem 8 applied to t_Z, t_n, t_g we obtain normalized term t'_Z, t'_n, t'_g which are provably (relative to $\widehat{\mathsf{WE\text{-}PA}}^\omega{\restriction}$) equal and such that every occurrence of $\mathcal{U}$ and K is of the form

$$\mathcal{U}(t[j^0]) \qquad \text{resp.} \qquad K(n^0, t[j^0]),$$

where t is a term in $T_0[\mathcal{U}, K, f]$.

Let $(t_i)_{i<n}$ be the list of all of these terms t to which $\mathcal{U}$ and K are applied. Assume that this list is partially ordered according to the subterm ordering, i.e. if t_i is a subterm of t_j then $i < j$.

We now build for each f a partial non-principal ultrafilter $\mathcal{F}$ which acts on these occurrences like a real non-principal ultrafilter. For this fix an arbitrary f.

The filter $\mathcal{F}$ is build by iterated applications of Proposition 6: To start the iteration let $\mathcal{A}_{-1}$ be the trivial algebra $\{\emptyset, \mathbb{N}\}$ and $\mathcal{F}_{-1} = \{\mathbb{N}\}$ be the partial non-principal ultrafilter for $\mathcal{A}_{-1}$.
Let $\mathcal{A}_i$ be the algebra spanned by $\mathcal{A}_{i-1}$ and the sets described by t_i where $\mathcal{U}, K$ are replaced by $\mathcal{F}_{i-1}$ and K' from (5), i.e. $\big(t_i[\mathcal{U}/\mathcal{F}_{i-1}, K/K'](j)\big)_{j\in\mathbb{N}}$. Let $\mathcal{F}_i$ be an extension of $\mathcal{F}_{i-1}$ to the new algebra $\mathcal{A}_i$ as constructed in Proposition 6.

Obviously in a term t_i the functional $\mathcal{F}$ is only applied to subterms of t_i. Since the (t_i) is sorted according to the subterm ordering the partial non-principal ultrafilter is already fixed for these applications. In other words, the theory proves that $t_i[\mathcal{U}/\mathcal{F}_{i-1}] = t_i[\mathcal{U}/\mathcal{F}_j]$ for every $j \geq i$.

For the resulting partial non-principal ultrafilter $\mathcal{F} := \mathcal{F}_n$ we then get

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \forall f\, \exists \mathcal{F}\, (\mathcal{U})_{qf}(\mathcal{F}, t_Z[\mathcal{F}, K', f], t_n[\mathcal{F}, K', f], K' t_n[\mathcal{F}, K', f] t_Z[\mathcal{F}, K', f]).$$

Combining this with (6) yields

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \forall f\, \exists \mathcal{F}\, \mathsf{A}_{qf}(f, t_g[\mathcal{F}, K', f])$$

and hence

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \forall f\, \exists g\, \mathsf{A}_{qf}(f, g).$$

With this we have eliminated the use of $(\mathcal{U})$ in the proof.

By Theorem 8.3.4 of [4] the theory $\mathsf{RCA}_0^\omega + (\mu)$ is conservative for sentences of this form over ACA_0^ω and therefore

$$\mathsf{ACA}_0^\omega \vdash \forall f\, \exists g\, \mathsf{A}_{qf}(f, g).$$

To obtain a realizer for g we apply the functional interpretation to the last statement. This extracts a realizer $t \in T_0[B_{0,1}]$ where $B_{0,1}$ is the bar recursor of lowest type, see Section 11.3 in [17]. Since t_g is only a term of type 2 one can find a term $t' \in T$ which is equal to t, see [15, Corollary 4.4.(1)]. This t' solves the theorem. □

If one is not interested in the extracted program then one can obtain a stronger conservation result:

Theorem 9 (Conservation)**.** *The system $\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U})$ is Π^1_2-conservative over ACA_0^ω and therefore also conservative over* PA*.*

Proof. Let $\forall f\, \exists g\, \mathsf{A}(f, g)$ be an arbitrary Π^1_2-statement which is provable in $\mathsf{ACA}_0^\omega + (\mu) + (\mathcal{U})$ and does not contain μ or $\mathcal{U}$. We will show that this statement is provable in ACA_0^ω and if it is arithmetical also in PA.

Relative to (μ) each arithmetical formula is equivalent to a quantifier free formula. Hence there exists a quantifier free formula A'_{qf} such that

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \mathsf{A}(f, g) \leftrightarrow \mathsf{A}'_{qf}(f, g).$$

This gives

$$\mathsf{RCA}_0^\omega + (\mu) + (\mathcal{U}) \vdash \forall f\, \exists g\, \mathsf{A}'_{qf}(f, g).$$

Since the system $\mathsf{RCA}_0^\omega + (\mu)$ has a functional interpretation in $T_0[\mu]$, see [4, 8.3.1], one can now apply the same argument as in the proof of Theorem 4 with T_0 replaced by $T_0[\mu]$, and obtains that

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \forall f\, \exists g\, \mathsf{A}'_{qf}(f, g)$$

and therefore also

$$\mathsf{RCA}_0^\omega + (\mu) \vdash \forall f\, \exists g\, \mathsf{A}(f, g).$$

The result follows now also from Theorem 8.3.4 of [4]. □

Remark 10. Let Φ be a set of sentences of the form $\forall x^1\, \mathsf{B}_{qf}(x)$. A careful inspection of the proof of the Theorems 4 and 9 shows that these remain true if one replaces ACA_0^ω with $\mathsf{ACA}_0^\omega + \Phi$ and adds Φ to the verifying theory in the case of Theorem 4. Furthermore, one may add provably extensional constants of degree ≤ 2.

Appendix A. Elimination of Skolem functions for monotone formulas

We will show in this appendix that uses of a partial non-principal ultrafilter for an algebra given by a fixed term over a weak basis theory does not lead to more than primitive recursive growth. For this we will make use of Kohlenbach's elimination of Skolem functions for monotone formulas, see [14], [17, Chapter 13].

Let WKL_0^* be the system WKL where Σ^0_1-IA is replaced by QF-IA and the exponential function and let $\mathsf{WKL}_0^{\omega *}$ be the corresponding finite type extension. For a formal definition see [20, X.4.1] and [16] for the finite type system.

Let Π^0_1-$\mathsf{CA}(f)$ be the restriction of Π^0_1-comprehension to the Π^0_1 formula given by f, i.e. the statement

$$\exists g \, \forall n \; (g(n) = 0 \leftrightarrow \forall x \, f(n,x) = 0)$$

Further, let $\mathcal{U}(\mathcal{A})$ be the principle that states that for the algebra $\mathcal{A} = (A_n)_{n \in \mathbb{N}}$ given by $(f(n))_{n\in\mathbb{N}}$ there exists a set $F \subseteq \mathbb{N}$, such that

$$\mathcal{F} = \{ A \mid \exists n \in F \, (A = A_n) \}$$

satisfies $(\mathcal{U})$ relativized to $\mathcal{A}$. This means that

$$\begin{cases} \quad \forall i,j \; \left(A_i = \overline{A_j} \rightarrow (i \in F \vee j \in F)\right) \\ \wedge \forall i,j \; ((A_i \subseteq A_j \wedge i \in F) \rightarrow j \in F) \\ \wedge \forall i,j,k \; ((i,j \in F \wedge A_k = A_i \cap A_j) \rightarrow k \in F) \\ \wedge \forall i \; (i \in F \rightarrow \forall n \, \exists k > n \, (k \in A_i)) \,. \end{cases}$$

We obtain the following theorem:

Theorem 11. *Let $\mathsf{A}_{qf}(f,x)$ be a quantifier free formula that contains only f, x free and let t_1, t_2 be terms in $\mathsf{WKL}_0^{\omega *}$. If*

$$\mathsf{WKL}_0^{\omega *} \vdash \forall f \; \left(\Pi^0_1\text{-}\mathsf{CA}(t_1 f) \wedge \mathcal{U}(t_2 f) \rightarrow \exists x \, \mathsf{A}_{qf}(f,x)\right)$$

then one can extract a primitive recursive (in the sense of Kleene) functional Φ such that

$$\mathsf{RCA}_0^\omega \vdash \forall f \, \mathsf{A}_{qf}(f, \Phi(f)).$$

In particular if f is only of type 0 one obtains that there exists a primitive recursive function g such that

$$\mathsf{PRA} \vdash \forall x \, \mathsf{A}_{qf}(x, g(x)).$$

Proof. We will show, by formalizing the construction of b in the proof of Proposition 6, that there exists a term t' such that

$$\forall h \; \left(\Pi^0_1\text{-}\mathsf{CA}(t'h) \rightarrow \mathcal{U}(h)\right) .$$

The theorem follows then from the elimination of Skolem functions for monotone formulas and the fact that one can code the two instances of Π^0_1-CA given by $t_1 f$ and $t' t_2 f$ into one. For the elimination of Skolem functions see for instance Proposition 13.20 in [17] — the statement of this proposition is essentially the same as of this theorem without $\mathcal{U}$. For the conservativity over PRA, see [4].

In the construction of b in the proof of Proposition 6 only two steps cannot be formalized in $\mathsf{WKL}_0^{\omega *}$. The first step is the application of Π^0_1-CP and the second is the use of Π^0_2-WKL. The use of Π^0_1-CP can be reduced to a suitable instance of Π^0_1-CA (with the parameters $\mathcal{F}, \tilde{\mathcal{A}}$) and QF-$\mathsf{AC}^{1,0}$. The use of Π^0_2-WKL follows from Π^0_1-WKL and another instance of Π^0_1-CA (also with the parameters $\mathcal{F}, \tilde{\mathcal{A}}$). Since

Π^0_1-WKL is equivalent to WKL and one can code the two instances of comprehension together one obtains in total that the index function b can be constructed in $\mathsf{WKL}_0^{\omega*} + \Pi^0_1$-$\mathsf{CA}(t\mathcal{F}\tilde{\mathcal{A}})$ for a suitable t. (Note that the set $\tilde{\mathcal{F}}$ cannot be defined since it involves μ.)

Using this one can extend the partial ultrafilter $\mathcal{F} = \{\mathbb{N}\}$ on the trivial algebra $\mathcal{A} = \{\emptyset, \mathbb{N}\}$ to an (index set of an) ultrafilter satisfying $\mathcal{U}(h)$. From this one can easily construct a term t'. This provides the theorem. ☐

Remark 12. Although the restriction of $\mathcal{U}$ to an algebra given by a term seems to be weak, it is strong enough to prove instances of ultralimit, i.e. that the ultralimit exists for (a sequence of) sequences or given by one fixed term.

To see this let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the interval $[0,1]$. Without loss of generality we may assume that $(x_n) \subseteq \mathbb{Q}$. We will prove that the ultralimit of this sequence exists using $\mathcal{U}(t[(x_n)])$ for a term t. For this let

$$A_{i,k} := \left\{ n \in \mathbb{N} \;\middle|\; x_n \in \left[\frac{i}{2^k}, \frac{i+1}{2^k}\right[\right\}.$$

Let $\mathcal{A}$ be the algebra created by this sets. It is clear that $\mathcal{A}$ can be described by a term $t[(x_n)]$.

Observe that the proof of Lemma 2 can also be carried out in RCA_0^*. Since $\left(A_{i,k}\right)_{i\leq 2^k}$ defines a finite partition of $\mathbb{N}$, Lemma 2 provides

$$\forall k\, \exists! i \leq 2^k\ (A_{i,k} \in \mathcal{U}),$$

(strictly speaking we obtain that the index of $A_{i,k}$ is in an index set of $\mathcal{U}$) and QF-$\mathsf{AC}^{1,0}$ yields a choice function $f(k)$ for i. Note that the ultrafilter properties provide that each $A_{f(k),k}$ is infinite and that

$$\forall k\, \forall k' > k\ \left(A_{f(k'),k'} \subseteq A_{f(k),k}\right).$$

Let $g(k)$ be the k-th element of $A_{f(k),k}$ then the sequence $\left(x_{g(k)}\right)_k$ defines a Cauchy-sequence with Cauchy-rate 2^{-k} which converges to $\lim_{n\to\mathcal{U}} x_n$.

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